RINGS OVER WHICH CYCLICS ARE DIRECT SUMS OF PROJECTIVE AND CS OR NOETHERIAN

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Dedicated to Patrick F. Smith on his 65th birthday.

Abstract. R is called a right WV-ring if each simple right R-module is injective relative to proper cyclics. If R is a right WV-ring, then R is right uniform or a right V-ring. It is shown that for a right WV-ring R, R is right noetherian if and only if each right cyclic module is a direct sum of a projective module and a CS or noetherian module. For a finitely generated module M with projective socle over a V-ring R such that every subfactor of M is a direct sum of a projective module and a CS or noetherian module, we show $M = X \oplus T$, where X is semisimple and T is noetherian with zero socle. In the case that $M = R$, we get $R = S \oplus T$, where S is a semisimple artinian ring, and T is a direct sum of right noetherian simple rings with zero socle. In addition, if R is a von Neumann regular ring, then it is semisimple artinian.

1. Introduction and Preliminaries

The question of studying homological properties on modules that guarantee the noetherian property dates back to the 1960s, when Bass and Papp showed that a ring is right noetherian iff direct sums of injective modules are injective. Since then, there has been continuous work on finding properties on classes of modules that guarantee the ring to be right noetherian (or some other finiteness condition). For instance, if each cyclic right module is: an injective module or a projective module [5], a direct sum of an injective module and a projective module [12, 14], or a direct sum of a projective module and a module Q, where Q is either injective or noetherian [6], then the ring is right noetherian. It is also known that if every finitely generated right module is CS, then the ring is right noetherian [7]. A celebrated theorem of Osofsky-Smith states that if every cyclic module is CS then R is a qfd-ring [12]. In this paper, we will consider rings over which every cyclic right module is a direct sum of a projective module and a CS or noetherian module.

In Section 2, we first introduce a slight generalization of V-rings, which we call WV-rings (weakly V-rings). Recall that rings over which all simple modules are injective are known as V-rings [11]. A ring R is called a right WV-ring if every simple right R-module is R/A-injective for any right ideal A of R such that $R/A \ncong R$. Detailed study as to how WV-rings differ from V-rings is provided in Section 2. Indeed, if R is a right WV-ring but not a right V-ring, then R must be right uniform.

In Section 3, we introduce the property (*) for an R-module M, and say that

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M satisfies (*) if we can write $M = A \oplus B$, where A is either a CS-module or a noetherian module, and B is a projective module. Theorem 18 (a) shows that if R is a V-ring and M is a finitely generated R-module with projective socle such that each subfactor of M satisfies (*), then $M = X \oplus T$, where X is semisimple and T is noetherian with zero socle. In particular, if R is a V-ring such that each cyclic module satisfies (*), then $R = S \oplus T$, where S is semisimple artinian and T is a finite direct sum of simple noetherian rings with zero socle. Theorem 18 (b) shows that for a WV-ring R, R is noetherian iff each cyclic R-module satisfies (*). The property (*) has been studied in [13] for finitely generated, as well as 2-generated, modules. The proofs of the main results depend upon a series of lemmas.

We will say a module has fud whenever it has finite uniform dimension. Throughout, we assume all rings are associative rings with identity, and all modules are right R-modules. Thus in our results, we shall omit the word 'right' when we want to say right noetherian, right WV-ring, etc. We shall use $\subset_e$ to denote an essential submodule, and $\subset_\oplus$ to denote a direct summand. For any undefined notation or terminology, we refer the reader to [9].

2. WV-rings

A ring R is called a WV-ring if each simple R-module is R/A- injective for any right ideal A such that $R/A \ncong R$ (i.e. R/A is proper). Such rings need not be V-rings, as for example the ring $\mathbb{Z}_{p^2}$ for any prime p is a WV-ring which is not a V-ring. Let us remark that Wisbauer ([16], p.190) called a module M co-semisimple if every module in the category $\sigma(M)$ is M-injective. Following Wisbauer's definition of co-semisimple modules, a right WV-ring is a ring for which every proper cyclic right module is co-semisimple.

Let us first compare V-rings and WV-rings.

Lemma 1. *Let R be a WV-ring, and R/A and R/B be proper cyclic modules such that $A \cap B = 0$. Then R is a V-ring.*

Proof: Since R is a WV-ring, any simple module is $R/A \times R/B$-injective. Since R_R embeds in $R/A \times R/B$, any simple module is R_R injective, i.e. R is a V-ring. □

Theorem 2. *Let R be a WV-ring which is not a V-ring. Then R must be uniform.*

Proof: Suppose R is a WV-ring. If R is of infinite uniform dimension, then R contains a direct sum $A \oplus B$ where both A and B are infinite direct sums of nonzero right ideals. If $R/A \cong R$, then R/A is projective, and hence there exists a right ideal C of R such that $R = C \oplus A$. But then the cyclic module R/C is isomorphic to an infinite direct sum of nonzero modules, a contradiction. Thus R/A is proper. Similarly R/B is proper, and so R is a V-ring by Lemma 1.

Assume now that $u.dim(R) = n > 1$ is finite. Then there exist closed uniform right ideals U_i such that $\bigoplus_{i=1}^n U_i \subset_e R$. Now $u.dim(R/U_1) = n-1 = u.dim(R/U_2)$, and so R/U_1 and R/U_2 are proper. Hence R is a V- ring by Lemma 1.

So if R is a WV-ring but not a V-ring, we must have $u.dim(R) = 1$, i.e. R is uniform. □

The proofs of the following propositions 3 and 5 are straightforward and follow closely the classical ones given in Lam ([9], Lemma 3.75 p.99) or Wisbauer ([16], 23.1 p.190).

Proposition 3. *Let R be a ring such that R/I is proper for any nonzero right ideal I. Then the following are equivalent:*

(a) R is a WV-ring.
(b) $J(R/I) = 0$ for any nonzero right ideal I.
(c) Any nonzero right ideal $I \neq R$ is an intersection of maximal right ideals.
(d) If a simple R-module is contained in a cyclic module R/I where $I \neq 0$, then it is a direct summand of R/I.
In particular, the above statements are equivalent when R is uniform or local. □

Corollary 4. *If R is a WV-ring, then $R/J(R)$ is a V-ring.*

Proof: Let $J = J(R)$. We note that R is a V-ring iff each right ideal ($\neq R$) is an intersection of maximal right ideals. If R is a WV-ring which is not uniform, then R is a V-ring (Theorem 2) and hence $J = 0$. So the result is clear in this case.

Thus we may assume R is uniform. By Proposition 3, every nonzero right ideal ($\neq R$) is an intersection of maximal right ideals. If $J = 0$, then the zero ideal is also an intersection of maximal right ideals, and so $R(= R/J)$ is a V-ring. If $J \neq 0$, then in R/J all right ideals ($\neq R/J$) are intersections of maximals, and so R/J is a V-ring. □

Proposition 5. *If R is a WV-ring, then the following statements hold:*

(a) If I is a right ideal of R, then either $I^2 = 0$ or $I^2 = I$.
(b) If R is a domain, then R is simple.
(c) If a nonzero right ideal I of R contains a nonzero two-sided ideal, then every simple R-module is R/I-injective.
(d) If R is a von Neumann regular ring, then R is a V-ring.
(e) If R is a local ring and is not a V-ring, then R has exactly three right ideals.

Proof: (a) If R is a V-ring, it is well known that $I^2 = I$ for every right ideal of R. Assume that R is not a V-ring. Then R is uniform (Theorem 2).

Let $I \neq R$ be a right ideal and suppose $I^2 \neq 0$. By Proposition 3, both I and I^2 are intersections of maximal right ideals. If $I^2 \neq I$, there must exist a maximal right ideal M such that $I^2 \subseteq M$ but $I \not\subseteq M$. We thus have $R = I + M$ and we can write $1 = x + m$ for some $x \in I, m \in M$. This gives $I = (x + m)I \subseteq xI + mI \subseteq I^2 + M = M$, a contradiction. Hence $I^2 = I$.

(b) Let $0 \neq a \in R$. Since R is a domain, $(aR)^2 \neq 0$, so part (a) gives us $(aR)^2 = aR$, i.e. $aRaR = aR$. Since R is a domain this gives that $RaR = R$.

(c) Let us first remark that if T is a nonzero two-sided ideal of R, then, since $(\frac{R}{T})T = 0$, R/T is a proper cyclic module.

Let I be a nonzero right ideal of R and T a nonzero two-sided ideal contained in I. Since R is a WV-ring, any simple module is R/T-injective, and hence any simple module is injective relative to $R/I \cong \dfrac{R/T}{I/T}$.

(d) Follows from Corollary 2.4, since $J(R) = 0$.

(e) If $I \neq 0$ and $I \neq R$, then I is an intersection of maximal right ideals (Proposition 3). So $I = J(R)$. Thus R has at most three right ideals. □

Corollary 6. *If R is a WV-domain, then R is a V-domain.* □

It is known that the property of R being a V-ring is not left-right symmetric, and hence neither is the property of being a WV-ring. In fact, the property of being a

WV- ring that is not a V-ring is not left-right symmetric either, as evidenced by the following example, due to Faith ([3], page 335).

Example 7. *Let* $R = \begin{bmatrix} a & b \\ 0 & \sigma(a) \end{bmatrix}$ *where* $a, b \in \mathbb{Q}(x)$ *and* σ *is the* $\mathbb{Q}$*-endomorphism of* $\mathbb{Q}(x)$ *such that* $\sigma(x) = x^2$*.*

In this ring there are only three left ideals and hence it is a left WV-ring. It cannot be a left V-ring because $J(R) \neq 0$. This ring is local and is not right noetherian and thus it cannot be a right WV-ring (Proposition 5 (e)).

3. Cyclics being (CS or Noetherian) $\oplus$ Projective

Recall that an R-module M satisfies (*) if we can write $M = A \oplus B$, where A is either a CS-module or a noetherian module, and B is a projective module. It was shown in [13] that a ring R is noetherian iff every 2-generated R-module satisfies (*). We remark that it is not sufficient to assume that every cyclic satisfies (*) in order for R to be noetherian, as may be seen from the following example [10].

Example 8. *Let* R *be the ring of all formal power series*

$$\{\sum a_i x^i | a_i \in F, i \in I\}$$

where F *is a field, and* I *ranges over all well-ordered sets of nonnegative real numbers.*

This ring is not noetherian, but every homomorphic image is self-injective, and hence satisfies (*).

We begin with a result that is used throughout the paper. We would like to thank a referee for drawing our attention to shock's result [15] which shortens the proof.

Proposition 9. *Let* C *be a cyclic* R*-module such that each cyclic subfactor of* C *satisfies (*), and let* $S = Soc(C)$*. Then* C/S *has* fud*. Furthermore, if* R *is a* WV*-ring, then* C/S *is noetherian.*

Proof: Let $E \subseteq_e C$ and $\frac{X}{D}$ be a cyclic subfactor of $\frac{C}{E}$, where $E \subseteq D \subseteq X \subseteq C$. Then by (*), $\frac{X}{D} = \frac{B}{D} \oplus \frac{A}{D}$ with $\frac{B}{D}$ projective and $\frac{A}{D}$ CS or noetherian. Since D splits from B, essentiality shows that $B = D$. Theorem 1.3 of [2] then applies to give that $\frac{C}{E}$ has fud. Since E was arbitrary such that $E \subseteq_e C$, this implies that, in particular that $\frac{C}{E}$ has qfd. Then $\frac{C}{soc(C)}$ is fud by Lemma 2.9 of [2]. Now assume that R is a WV-ring and let $Z \subset Y \subseteq C/E$, where, as above, $E \subseteq_e C$. If $0 \neq x \in rad(Y/Z)$ let K be maximal in xR. Since $\frac{C}{E}$ is singular it is proper cyclic and the simple module $\frac{xR}{K}$ is $\frac{C}{E}$-injective, so it splits in $\frac{Y/Z}{K}$, a contradiction since $\frac{xR}{K} \subseteq rad(\frac{Y/Z}{K})$. We conclude that $rad(\frac{Y}{Z}) = 0$. Then Theorem 3.8 of [15] implies that $\frac{C}{E}$ is noetherian. Now, Theorem 5.15 (1) in [1] shows that $\frac{C}{soc(C)}$ is noetherian. □

For the convenience of the reader, we state below a well-known lemma (Cf. Lemma 9.1 p. 73 [1]) .

Lemma 10. *If* M *is a finitely generated* CS*-module and* $\oplus M_i$ *is an infinite direct sum of nonzero submodules of* M*, then* $M/\oplus M_i$ *cannot have finite uniform dimension.* □

Under a stronger assumption on a cyclic module C than the condition (*), namely, if every cyclic subfactor of C is projective, CS, or noetherian, we show that C is noetherian when R is a WV-ring. This will play a key role later as we work towards the general result.

Theorem 11. *Let C be a cyclic R-module such that each cyclic subfactor of C is either CS, noetherian, or projective.*
(a) Then C has fud.
(b) If R is a WV-ring, then C is noetherian.

Proof: (a) Let $S = Soc(C)$. By Proposition 9, C/S has fud. We show S is finitely generated.

Suppose S is infinitely generated. Write $S = S_1 \oplus S_2$, where S_1 and S_2 are both infinitely generated. Now by hypothesis, C/S_1 is either CS or noetherian or projective. If projective, then $S_1 \subset_\oplus C$ and hence S_1 is cyclic, a contradiction as S_1 is infinitely generated. If noetherian, then so is $S/S_1 \cong S_2$, a contradiction as S_2 is infinitely generated. So C/S_1 is CS. Furthermore, $(S_1 \oplus S_2)/S_1 \cong S_2$ is infinitely generated. Since $C/S \cong \dfrac{C/S_1}{(S_1 \oplus S_2)/S_1}$, we get a contradiction by invoking Lemma 10. Hence S is finitely generated and so C has fud.

(b) Since C/S is noetherian (Proposition 9), and S has fud, it follows that C is noetherian. $\square$

Although the proof of the next lemma is straightforward we believe that the reader will appreciate the simple technique used in the proof.

Lemma 12. *Let C be an R-module and $S = Soc(C)$. If C/S is a uniform R-module, then for any two submodules A and B of C with $A \cap B = 0$, either A or B is semisimple.*

Proof: Let K be a complement submodule of A in C containing B. Then $A \oplus K \subset_e C$. This yields $Soc(A \oplus K) = S$. Thus, $(A \oplus K)/(Soc(A \oplus K)) \subseteq C/S$. Since $(A \oplus K)/(Soc(A \oplus K)) \cong A/Soc(A) \times K/Soc(K)$ and C/S is uniform as an R-module, either $A/Soc(A)$ or $K/Soc(K)$ is zero. So $A = Soc(A)$ or $K = Soc(K)$. In other words, either A or K (and hence B) is semisimple. $\square$

Lemma 13. *If C is an R-module, and if $C/I = A/I \oplus B/I$ is a direct sum with B/I a projective module, then $C = A \oplus B'$, where $B = B' \oplus I$.*

Proof: From the decomposition $C/I = A/I \oplus B/I$, we have $C = A + B$, where $A \cap B = I$. Since B/I is projective, $B = B' \oplus I$ for some B'. Then $C = A + (B' \oplus I) = A + B'$. We claim that $A \cap B' = 0$.

Let $x \in A \cap B' \subseteq A \cap B = I$. Then $x \in B' \cap I = 0$. Thus $C = A \oplus B'$. $\square$

Lemma 14. *Let R be a WV-ring. Let C be a cyclic module with a projective socle (equivalently $S = Soc(C)$ is embeddable in R). If C/S is a uniform R-module and each cyclic subfactor of C satisfies (*), then C is noetherian.*

Proof: First assume R is a V-ring. Let C'/I be a cyclic subfactor of C and write $C'/I = A/I \oplus B/I$ as a direct sum of a CS or noetherian module and a projective module, respectively. Then, by Lemma 13, $C' = A \oplus B'$, where $B = B' \oplus I$. Since $\frac{C'}{Soc(C')} = \frac{C'}{C' \cap S} \cong \frac{C'+S}{S} \subseteq \frac{C}{S}$ is uniform, either A or B' is semisimple (Lemma 12).

Note both A and B' are cyclic.

Case 1: A is semisimple. Since A/I is semisimple cyclic and R is a V-ring, A/I is injective. Moreover A/I embeds in $A \subseteq Soc(C') \subseteq Soc(C) = S$. The hypothesis that S embeds in R yields that A/I and hence C'/I are projective.

Case 2: B' is semisimple. Since $B/I \cong B'$ is semisimple and cyclic, it is a finite direct sum of simple injective modules. If A/I is noetherian, then clearly C/I will be also. Recall that a direct sum of a CS module and a simple module is a CS module (Cf. Lemma 7.10 [1]). Hence if A/I is CS then $A/I \oplus B/I$ is also CS.

Thus, any cyclic subfactor of the cyclic module C is either CS or noetherian or projective. Therefore, C is noetherian by Theorem 11.

Now if R is not a V-ring, then it is uniform (Theorem 2). Since S embeds in R, S is trivially noetherian. So C is noetherian because C/S is noetherian (Proposition 9). $\square$

Proposition 15. *Let R be a V-ring. Let C be a cyclic R-module with essential and projective socle. Suppose each cyclic subfactor of C satisfies (*). Then C is semisimple.*

Proof: Let $S = Soc(C)$. We know by Proposition 9 that C/S has fud. Suppose $C/S \neq 0$. Then C/S contains a nonzero cyclic uniform submodule. Thus we can find $u \in C$ with $U = (uR + S)/S \cong uR/Soc(uR)$ uniform. Since $Soc(uR) \subseteq_{\oplus} S$ we know that $Soc(uR)$ is projective. Moreover, every cyclic subfactor of uR also satisfies (*). Thus Lemma 14 implies that uR is noetherian. $Soc(uR)$ is then a finite direct sum of simple modules, and hence it is injective. Since $Soc(uR) \subset_e uR$, $uR = Soc(uR)$. This yields $U = 0$, a contradiction. Thus $C/S = 0$, that is, $C = S$, completing the proof. $\square$

Remark 16. *We note that the above proposition does not apply to a WV-ring R which is not a V-ring. In this case R_R is uniform and the only projective submodule of a cyclic R-module is the zero submodule.*

Theorem 17. *Let R be a von Neumann regular WV-ring such that each cyclic R-module satisfies (*). Then R is semisimple artinian.*

Proof: By Proposition 5 (d), R is a V-ring. Let $S = Soc(R)$. Then R/S has fud and hence it is semisimple artinian [8]. Let T be a complement of S. Then T embeds essentially in R/S. Thus $T = 0$. Hence $S \subset_e R$. So by Proposition 15, R is semisimple artinian. $\square$

Finally, we prove the following general result.

Theorem 18. *(a) Let R be a V-ring. Let M be a finitely generated R-module with projective socle. Suppose each cyclic subfactor of M satisfies (*). Then M is noetherian, and $M = X \oplus T$, where X is semisimple and T is noetherian with zero socle.*
In particular, if R is a V-ring such that each cyclic module satisfies (*), then $R = S \oplus T$, where S is semisimple artinian and T is a finite direct sum of simple noetherian rings with zero socle.
(b) For a WV-ring R, R is noetherian iff each cyclic R-module satisfies (*).

Proof: (a) First, assume M is cyclic. Let $S_0 = Soc(M)$ and let T_0 be a complement of S_0 in M. Consider the cyclic module $X_0 = M/T_0$. Then S_0 is essentially embeddable in X_0. Since $Soc(X_0) \cong S_0$, X_0 is semisimple by Proposition 15. So X_0, and hence S_0, is a finite direct sum of simples. In particular S_0 is injective and we have $M = S_0 \oplus T_0$. Since M/S_0 is noetherian (Proposition 9), T_0 is noetherian and obviously it has zero socle.

In general, $M = \sum_{i=1}^{n} x_i R$. By above, each $x_i R$ is noetherian, and hence M is noetherian. $X = Soc(M)$ is finitely generated and injective by hypothesis. Therefore $M = X \oplus T$, where X is semisimple and T is noetherian with zero socle.

Finally, let $S = Soc(R)$ which is clearly projective in a V-ring and let T be its complement. Then, as shown above, R is a right noetherian V-ring. Therefore, R is a direct sum of simple noetherian rings ([4], page 70). So, $R = S \oplus T$, where S is semisimple artinian and T is a finite direct sum of simple noetherian rings with zero socle.

(b) Note that if R is a WV-ring and not a V-ring, then R is uniform (Theorem 2). In this case, $Soc(R)$ is either zero or a minimal right ideal. Since $R/Soc(R)$ is noetherian (Proposition 9), we conclude that R is noetherian. The converse is obvious. $\square$

Remark 19. *(a) Although Theorem 17 is a consequence of Theorem 18, the short proof given for Theorem 17 is of independent interest. More generally, if R is a WV -ring in which each non-nil right ideal contains a nonzero idempotent and every cyclic R-module satisfies (*), then R is semisimple artinian* [8].

(b) Readers familiar with the Wisbauer Category $\sigma[M]$ may observe that the results in this paper can be more generally stated in $\sigma[M]$, where M is a finitely generated module.

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